

# Averages of short correlations: a note

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## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS.

We have studied the so-called “CORRELATIONS” (mainly, in particular case  $g = f$  of *autocorrelations*) of two arithmetic functions  $f, g : \mathbb{N} \rightarrow \mathbb{C}$ , that are also called (much more, in the modular forms environment) “*shifted convolution sums*”, i.e. (in this paper we do not conjugate  $g(n - a)$ , even if  $g$  values are complex)

$$\mathcal{C}_{f,g}(N, a) \stackrel{\text{def}}{=} \sum_{n \sim N} f(n)g(n - a),$$

where  $a$  is called the SHIFT, the abbreviation  $n \sim N$  in sums stands for  $N < n \leq 2N$ , for integers  $N > 0$  (more generally,  $n \sim X$  is  $X < n \leq 2X$ , for real  $X > 0$ ). Actually, our “Generations”, to abbreviate [CL1] title, displays, with different approaches for the asymptotic and upper bound estimates, many different types of averages, for such quantities; mainly, it deals with three generations, namely three kinds of averages starting with the easiest, i.e., what we call the FIRST GENERATION of correlation averages:

$$\sum_{a \leq H} \mathcal{C}_f(a),$$

where we abbreviate  $a \leq H$  for  $1 \leq a \leq H$  and  $\mathcal{C}_f(a)$  for  $\mathcal{C}_{f,f}(N, a)$ , in above notation (in fact, an autocorrelation). Hereafter we stick to *short interval* averages, namely  $H = o(N)$ , when  $N \rightarrow \infty$ . In the following,  $h \ll HN^{-\varepsilon}$  will appear, as a kind of length for “super-short intervals”. (Compare our main results.) Hereafter the Vinogradov notation  $\ll$  and the synonymous Landau’s  $O$  notation will be used; also, with subscripts to indicate the  $O$ -constant dependence. (Typically,  $\ll_{\varepsilon}$  depends on arbitrarily small  $\varepsilon > 0$ .)

In our [CL1] study and in the subsequent [CL2] it is clear that the correlations of shift  $a$  are immediately linked (see [E], for a great exposition) to the arithmetic progressions, with residue class  $a$  :

$$\mathcal{C}_{f,g}(N, a) = \sum_{q \leq Q} g'(q) \sum_{\substack{n \sim N \\ n \equiv a \pmod{q}}} f(n),$$

assuming that the so-called ERATOSTHENES TRANSFORM of our  $g$ ,  $g' \stackrel{\text{def}}{=} g * \mu$  (with  $*$  the Dirichlet product and  $\mu$  Möbius arithmetic function), has support up to  $Q$  (see the following). Since we clearly assumed to have  $Q \ll N$ , there’s no problem with the number of such  $n$ , in the inner sum (in worst case  $\gg 1$  of them).

In this paper we study the (a priori) much harder problem of estimating sums of the kind (here  $H \geq h$  and  $H = o(x)$ , as we may assume  $N \ll x \ll N$ ):

$$\sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n - a) = \sum_{a \leq H} \sum_{q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n)$$

where the “short” length  $h \leq H$ , instead of previous “long” length  $N$ , renders this kind of *first generation for short correlations* much more difficult (at least “for arithmetic progressions”, compare Theorem 0), as

$$q > h \implies \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) = \begin{cases} f(n_a) & \text{if } n_a \in (x, x+h] \\ 0 & \text{otherwise} \end{cases}$$

(see the proof of Lemma in §2) is involving “*sporadic sums*”, i.e. sums with “*at most one term and*”, in a probabilistic sense, “*vanishing most of the time*”. (In the above example, see Lemma proof, this simply means : the number of  $a$ s for which the  $n$ -sum doesn’t vanish turns out to be “small”, compared to all  $a$ s.)

However, an elementary argument (our Lemma) shows that we may gather  $H \geq h$  residue classes, so to avoid sporadically appearing terms ! Here, the fact that we consider a kind of average, instead of single correlations, is vital ! (Actually, without any kind of average, short correlations may not be treated at all !)

Before going on, we need the following two definitions (for which, compare [CL1] & [CL2]).

We say, for a general arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$ ,

$$f \text{ is ESSENTIALLY BOUNDED} \stackrel{\text{def}}{\iff} \forall \varepsilon > 0, f(n) \ll_{\varepsilon} n^{\varepsilon}, \text{ as } n \rightarrow \infty.$$

(Notice : compare this with Ramanujan's Conjecture in Selberg Class and in modular forms environment.)

We say  $f : \mathbb{N} \rightarrow \mathbb{C}$  is a SIEVE FUNCTION OF RANGE  $Q$  when the Eratosthenes transform of  $f$ , say,  $f' : \mathbb{N} \rightarrow \mathbb{C}$  (so that  $f = f' * \mathbf{1}$  by Möbius inversion : hereafter  $\mathbf{1}$  is the constant-1 arithmetic function), is supported up to  $Q$  and essentially bounded; in other words, the sieve functions of range  $Q$  are representable as

$$f(n) \stackrel{\text{def}}{=} \sum_{q|n, q \leq Q} f'(q),$$

for a certain arithmetic function  $f'$  (the  $f$  Eratosthenes transform), satisfying  $f'(q) \ll_{\varepsilon} q^{\varepsilon}$ . Since, by Möbius inversion, “ $f$  is essentially bounded” is equivalent to “ $f * \mu$  is essentially bounded”, we may also identify sieve functions of range  $Q$  with truncated divisor sums up to  $Q$  that satisfy Ramanujan Conjecture.

We give, for first, our easiest result for averages of short correlations.

**Theorem 0.** *Let  $h, H, x \in \mathbb{N}$  with  $h \leq H$  and  $H = o(x)$ , as  $x \rightarrow \infty$ . Take any ESSENTIALLY BOUNDED arithmetic functions  $f, g : \mathbb{N} \rightarrow \mathbb{C}$ . Then*

$$\sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) = \left( \sum_{a \leq H} g(x-a) \right) \left( \sum_{x < n \leq x+h} f(n) \right) + O_{\varepsilon}(x^{\varepsilon} h^2).$$

**Proof** is exhibited here, as it is 1-line (setting  $b := a - n + x$  and back  $a$  instead of  $b$ , using  $n - x \ll h$ ) :

$$\sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) = \sum_{x < n \leq x+h} f(n) \sum_{a \leq H} g(x - (a - n + x)) = \sum_{x < n \leq x+h} f(n) \sum_{a \leq H} g(x-a) + O_{\varepsilon}(x^{\varepsilon} h^2).$$

□

(A symbol □ will signify, as usual, the end of a Proof, likewise ◇ the end of a Remark.)

**Remark 0.** The remainders are non-trivial when  $h \ll H/x^{\varepsilon}$ , i.e., the  $H$ -average is “long enough”. ◇

**Remark 1.** See that the following results for short correlations averages are intended both to give some further insight and to get new results, from comparing with Theorem 0; which is, of course, far simpler ! ◇

Our second main result for averages of short correlations will be proved in §3.

**Theorem 1.** *Let  $h, H, Q, x \in \mathbb{N}$  with  $h \leq H < Q$  and  $Q \ll x$ ,  $H = o(x)$  as  $x \rightarrow \infty$ . Take any ESSENTIALLY BOUNDED  $f : \mathbb{N} \rightarrow \mathbb{C}$  and a SIEVE FUNCTION  $g : \mathbb{N} \rightarrow \mathbb{C}$  OF RANGE  $Q$  with Eratosthenes transform  $g'$ . Then*

$$\begin{aligned} \sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) &= \left( H \sum_{q \leq h} \frac{g'(q)}{q} + \sum_{h < q \leq H} g'(q) \left[ \frac{H}{q} \right] + \sum_{\substack{h < q \leq Q \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + \\ &\quad + O_{\varepsilon}(x^{\varepsilon} h^2). \end{aligned}$$

See, for remarks and comments, soon after its proof.

Rather an identical argument also gives the following companion result, proved in §3, too.

**Theorem 2.** *Let the same hypotheses of Theorem 1 hold. Then*

$$\begin{aligned} \sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) &= \left( \sum_{a \leq H} g(x-a) \right) \left( \sum_{x < n \leq x+h} f(n) \right) + O_\varepsilon(x^\varepsilon h^2) + \\ &+ \left( H \sum_{q \leq h} \frac{g'(q)}{q} \right) \sum_{x < n \leq x+h} f(n) + \sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) - \left( \sum_{a \leq H} \sum_{\substack{q \leq H \\ q|x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n). \end{aligned}$$

In passing, from the comparison of Theorem 0 and Theorem 2, the following result immediately follows.

**Corollary.** *Let  $h, H, x \in \mathbb{N}$  with  $h \leq H$  and  $H = o(x)$ , as  $x \rightarrow \infty$ . Take any ESSENTIALLY BOUNDED arithmetic functions  $f, g' : \mathbb{N} \rightarrow \mathbb{C}$ . Then*

$$\left( H \sum_{q \leq h} \frac{g'(q)}{q} \right) \sum_{x < n \leq x+h} f(n) + \sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) - \left( \sum_{a \leq H} \sum_{\substack{q \leq H \\ q|x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n) \ll_\varepsilon x^\varepsilon h^2.$$

This entails, choosing first a vanishing  $g'$  in the interval  $[1, h]$  and, then, vanishing in  $(h, H]$ , BOTH

$$\sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \left( \sum_{a \leq H} \sum_{\substack{h < q \leq H \\ q|x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2)$$

AND

$$\left( \sum_{a \leq H} \sum_{\substack{q \leq h \\ q|x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n) = \left( H \sum_{q \leq h} \frac{g'(q)}{q} \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2).$$

**Remark 2.** The further hypotheses on  $g$  and  $Q$ , from Theorem 2 (i.e. from Th.m 1), are not used at all !  $\diamond$

As we saw before, it is then immediate to glue together these *super-short intervals* to get, as a corollary to Theorem 0, a result for the first generation of correlations (these are the “long ones”, this time).

**Theorem 3.** *Let  $H, D, Q, N \in \mathbb{N}$  with  $H < Q$  and  $D, Q \ll N$ ,  $H = o(N)$  as  $N \rightarrow \infty$ . Choose an integer  $h$  such that  $h \rightarrow \infty$  as  $N \rightarrow \infty$ . Take two SIEVE FUNCTIONS  $f, g : \mathbb{N} \rightarrow \mathbb{C}$  OF RANGES  $D, Q$ , resp. Then*

$$\sum_{a \leq H} \sum_{n \sim N} f(n)g(n-a) = \sum_{a \leq H} \sum_{n \sim N} f(n)g(h[n/h] - a) + O_\varepsilon \left( N^\varepsilon H \left( \frac{Nh}{H} + \frac{N}{h} + h \right) \right).$$

Furthermore, averaging up to such a fixed integer,

$$\sum_{a \leq H} \sum_{n \sim N} f(n)g(n-a) = \sum_{a \leq H} \sum_{n \sim N} f(n) \frac{1}{h} \sum_{m \leq h} g(m[n/m] - a) + O_\varepsilon \left( N^\varepsilon H \left( \frac{Nh}{H} + \frac{N}{h} + h \right) \right).$$

**Remark 3.** The second formula follows by estimating trivially for  $m \leq L := \log N \rightarrow \infty$  and for  $L < m \leq h$  applying first formula (all “logs” and so on are inside  $N^\varepsilon$ ), whose proof we leave to the interested reader.  $\diamond$

We’ll study applications of Theorem 3 in future papers (and versions of present one).

The paper is organized as follows: our Lemma is given and proved in §2, then §3 proves main results, namely Theorem 1 and Theorem 2. (Notice that we call these “main”, as opposed to side results, i.e., the Lemma and the Remarks. These, in turn, have their own interest, not only in view of proving Theorems.)

## 2. PINCH LEMMA.

We start with so-called “PINCH LEMMA”, since it pinches, so to speak, integers in (very) short intervals, “squeezing” them to the left interval extreme (see Remark 4, after the proof). In fact, these short intervals contain at most one integer in the specified residue class modulo  $q$ , since  $q > h$ , where  $h$  is the interval’s length. We actually rebuild the whole interval, after summing over  $H \geq h$  residue classes.

**Lemma.** *Let  $h, H, Q \in \mathbb{N}$  with  $h \leq H < Q$  and  $Q \ll x$ ,  $H = o(x)$ , as  $x \rightarrow \infty$ . Take two ESSENTIALLY BOUNDED arithmetic functions  $f, g' : \mathbb{N} \rightarrow \mathbb{C}$ . Then BOTH*

$$\sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) = \left( \sum_{h < q \leq H} g'(q) \left\lfloor \frac{H}{q} \right\rfloor + \sum_{\substack{h < q \leq H \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2)$$

AND

$$\sum_{a \leq H} \sum_{H < q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) = \left( \sum_{\substack{H < q \leq Q \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2).$$

Notice we have non-triviality for the remainders when  $h$  is significantly smaller than  $H$ , say  $\log h < \log H$ .

**Proof.** For the first formula write LHS (left hand side) as

$$\begin{aligned} & \sum_{h < q \leq H} g'(q) \left( \sum_{a \leq q \lfloor \frac{H}{q} \rfloor} + \sum_{q \lfloor \frac{H}{q} \rfloor < a \leq H} \right) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) = \\ & = \left( \sum_{h < q \leq H} g'(q) \left\lfloor \frac{H}{q} \right\rfloor \right) \sum_{x < n \leq x+h} f(n) + \sum_{h < q \leq H} g'(q) \sum_{a \leq q \{\frac{H}{q}\}} \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) \end{aligned}$$

(recall  $a \leq A$  is  $1 \leq a \leq A$ ) and see that  $0 < a \leq q\{H/q\} < q$  allows to write

$$\sum_{a \leq q \{\frac{H}{q}\}} \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} f(n) = \sum_{a \leq q \{\frac{H}{q}\}} \sum_{\frac{x-a}{q} < m \leq \frac{x-a+h}{q}} f(qm + a),$$

whence  $m$ -sum is SPORADIC (i.e., contains at most one term, as the interval has length  $h/q < 1$ ) and writing the  $m$ -interval as

$$\left( \frac{x-a}{q}, \frac{x-a+h}{q} \right] = \left( \left\lfloor \frac{x}{q} \right\rfloor + \left\{ \frac{x}{q} \right\} - \frac{a}{q}, \left\lfloor \frac{x}{q} \right\rfloor + \left\{ \frac{x}{q} \right\} - \frac{a}{q} + \frac{h}{q} \right]$$

we see that  $m = \lfloor x/q \rfloor$  or  $m = \lfloor x/q \rfloor + 1$  (in particular,  $H = o(x) \Rightarrow qm > x - a > 0 \Rightarrow m > 0$ ), therefore (observe: next  $a$ -intervals are disjoint)

$$(*) \quad \sum_{\frac{x-a}{q} < m \leq \frac{x-a+h}{q}} f(qm + a) = f\left(q \left\lfloor \frac{x}{q} \right\rfloor + a\right) \mathbf{1}_{q\{\frac{x}{q}\} < a \leq q\{\frac{x}{q}\} + h} + f\left(q \left\lfloor \frac{x}{q} \right\rfloor + q + a\right) \mathbf{1}_{a \leq q\{\frac{x}{q}\} + h - q}$$

gives

$$\sum_{a \leq q \{\frac{H}{q}\}} \sum_{\frac{x-a}{q} < m \leq \frac{x-a+h}{q}} f(qm + a) = \sum_{\substack{a \leq q \{\frac{H}{q}\} \\ q\{\frac{x}{q}\} < a \leq q\{\frac{x}{q}\} + h}} f\left(q \left\lfloor \frac{x}{q} \right\rfloor + a\right) + \sum_{\substack{a \leq q \{\frac{H}{q}\} \\ a \leq q\{\frac{x}{q}\} + h - q}} f\left(q \left\lfloor \frac{x}{q} \right\rfloor + q + a\right) =$$

$$= \mathbf{1}_{\{\frac{x}{q}\} \leq \{\frac{H}{q}\} - \frac{h}{q}} \sum_{x < n \leq x+h} f(n) + \mathbf{1}_{\{\frac{H}{q}\} - \frac{h}{q} < \{\frac{x}{q}\} < \{\frac{H}{q}\}} O_\varepsilon(x^\varepsilon h) + \mathbf{1}_{\{\frac{x}{q}\} > 1 - \frac{h}{q}} O_\varepsilon(x^\varepsilon h).$$

Here we abbreviate  $\mathbf{1}_\varphi \stackrel{\text{def}}{=} 1$ , when  $\varphi$  is true,  $\stackrel{\text{def}}{=} 0$  otherwise (while  $\mathbf{1}$  is the constant-1 arithmetic function). Once summed over  $h < q \leq H$  with  $g'(q)$ , it implies first formula, because:

$$\begin{aligned} & \sum_{h < q \leq H} g'(q) \sum_{a \leq q \{\frac{H}{q}\}} \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \left( \sum_{\substack{h < q \leq H \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\} - \frac{h}{q}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + \\ & + O_\varepsilon(x^\varepsilon \sum_{\substack{h < q \leq H \\ \{\frac{H}{q}\} - \frac{h}{q} < \{\frac{x}{q}\} < \{\frac{H}{q}\}}} h) + O_\varepsilon(x^\varepsilon \sum_{\substack{h < q \leq H \\ 1 - \frac{h}{q} < \{\frac{x}{q}\} < 1}} h) = \left( \sum_{\substack{h < q \leq H \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2), \end{aligned}$$

since the DIVISOR FUNCTION is essentially bounded :  $\mathbf{d}(n) \stackrel{\text{def}}{=} \sum_{d|n} 1 \ll_\varepsilon n^\varepsilon$ , a well-known fact we use inside BOTH

$$\begin{aligned} \sum_{\substack{h < q \leq H \\ \{\frac{H}{q}\} - \frac{h}{q} < \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} 1 &= \sum_{\substack{h < q \leq H \\ q \{\frac{H}{q}\} - h < q \{\frac{x}{q}\} \leq q \{\frac{H}{q}\}}} 1 = \sum_{h < q \leq H} \sum_{\substack{0 \leq a < h \\ a = q \{\frac{H}{q}\} - q \{\frac{x}{q}\}}} 1 = \sum_{h < q \leq H} \sum_{\substack{0 \leq a < h \\ x+a \equiv H \pmod q}} 1 = \\ &= \sum_{0 \leq a < h} \sum_{\substack{h < q \leq H \\ q|x+a-H}} 1 \leq \sum_{0 \leq a < h} \mathbf{d}(x+a-H) \ll_\varepsilon x^\varepsilon h \end{aligned}$$

(here and in the following we use that  $a$ -intervals have length  $\leq h < q$ ) AND

$$\sum_{\substack{h < q \leq H \\ 1 - \frac{h}{q} < \{\frac{x}{q}\} < 1}} 1 = \sum_{\substack{h < q \leq H \\ q - h < q \{\frac{x}{q}\} < q}} 1 = \sum_{h < q \leq H} \sum_{\substack{0 < a < h \\ a = q - q \{\frac{x}{q}\}}} 1 = \sum_{0 < a < h} \sum_{\substack{h < q \leq H \\ q|x+a}} 1 \leq \sum_{0 < a < h} \mathbf{d}(x+a) \ll_\varepsilon x^\varepsilon h.$$

(For these and following bounds, it is *vital* that the  $\mathbf{d}(n)$  has  $n > 0$  : thanks to  $H = o(x)$  hypothesis.) Analogously, for the second formula, we apply  $(*)$  to get (see that now  $a < q$  is for free from  $H < q$ )

$$\begin{aligned} & \sum_{a \leq H} \sum_{H < q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \sum_{H < q \leq Q} g'(q) \sum_{a \leq H} \sum_{\substack{\frac{x-a}{q} < m \leq \frac{x-a+h}{q}}} f(qm+a) = \\ &= \sum_{H < q \leq Q} g'(q) \sum_{\substack{a \leq H \\ q \{\frac{x}{q}\} < a \leq q \{\frac{x}{q}\} + h}} f(qm+a) + O_\varepsilon(x^\varepsilon \sum_{\substack{H < q \leq Q \\ 1 - \frac{h}{q} < \{\frac{x}{q}\} < 1}} h) = \\ &= \sum_{\substack{H < q \leq Q \\ q \{\frac{x}{q}\} \leq H-h}} g'(q) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h \sum_{\substack{H < q \leq Q \\ H-h < q \{\frac{x}{q}\} < H}} 1) + O_\varepsilon(x^\varepsilon h^2) = \\ &= \sum_{\substack{H < q \leq Q \\ q \{\frac{x}{q}\} \leq H}} g'(q) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h \sum_{\substack{H < q \leq Q \\ H-h < q \{\frac{x}{q}\} \leq H}} 1) + O_\varepsilon(x^\varepsilon h^2), \end{aligned}$$

similarly as above for  $q-h < q\{x/q\} < q$  and we conclude by the analogous (set  $a = H - q\{x/q\}$  now)

$$\sum_{\substack{H < q \leq Q \\ H-h < q \{\frac{x}{q}\} \leq H}} 1 = \sum_{0 \leq a < h} \sum_{\substack{H < q \leq Q \\ q|x+a-H}} 1 \leq \sum_{0 \leq a < h} \mathbf{d}(x+a-H) \ll_\varepsilon x^\varepsilon h.$$

□

**Remark 4.** We may write Lemma's second formula as (we pinch  $n \equiv a \pmod q$  so it "becomes"  $x \equiv a \pmod q$ )

$$\sum_{a \leq H} \sum_{H < q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \left( \sum_{a \leq H} \sum_{\substack{H < q \leq Q \\ q|x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon(x^\varepsilon h^2).$$

◇

### 3. PROOF OF MAIN RESULTS.

**Proof (Th.1).** Taking  $g'$  in the Pinch Lemma to be the Eratosthenes Transform of  $g$ , open  $g(n-a)$  :

$$\begin{aligned} \sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) &= \sum_{a \leq H} \sum_{q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \sum_{a \leq H} \left( \sum_{q \leq h} + \sum_{h < q \leq H} + \sum_{H < q \leq Q} \right) g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) \\ &= I + II + III, \end{aligned}$$

say, where

$$\begin{aligned} I &:= \sum_{a \leq H} \sum_{q \leq h} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \sum_{q \leq h} g'(q) \left( \sum_{a \leq q \leq \frac{H}{q}} + \sum_{q \leq \frac{H}{q} < a \leq H} \right) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \\ &= \sum_{q \leq h} g'(q) \left[ \frac{H}{q} \right] \sum_{x < n \leq x+h} f(n) + O_\varepsilon \left( x^\varepsilon h \sum_{x < n \leq x+h} 1 \right) = H \sum_{q \leq h} \frac{g'(q)}{q} \sum_{x < n \leq x+h} f(n) + O_\varepsilon \left( x^\varepsilon h \sum_{x < n \leq x+h} 1 \right) = \\ &= \left( H \sum_{q \leq h} \frac{g'(q)}{q} \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon (x^\varepsilon h^2), \end{aligned}$$

while by the Pinch Lemma

$$II := \sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \left( \sum_{h < q \leq H} g'(q) \left[ \frac{H}{q} \right] + \sum_{\substack{h < q \leq H \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon (x^\varepsilon h^2)$$

and

$$\begin{aligned} III &:= \sum_{a \leq H} \sum_{H < q \leq Q} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) = \left( \sum_{\substack{H < q \leq Q \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon (x^\varepsilon h^2) = \\ &= \left( \sum_{\substack{H < q \leq Q \\ \{\frac{x}{q}\} \leq \{\frac{H}{q}\}}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + O_\varepsilon (x^\varepsilon h^2). \end{aligned}$$

□

**Remark 5.** However, see the definition of  $II$  in the Theorem proof, same hypotheses give

$$\begin{aligned} \sum_{a \leq H} \sum_{x < n \leq x+h} f(n)g(n-a) &= \left( H \sum_{q \leq h} \frac{g'(q)}{q} + \sum_{a \leq H} \sum_{\substack{H < q \leq Q \\ q | x-a}} g'(q) \right) \sum_{x < n \leq x+h} f(n) + \\ &+ \sum_{a \leq H} \sum_{h < q \leq H} g'(q) \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod q}} f(n) + O_\varepsilon (x^\varepsilon h^2), \end{aligned}$$

thanks to Remark 4 (after Pinch Lemma proof), too.

◇

**Proof (Th.2).** Follows from previous Remark and

$$\sum_{\substack{H < q \leq Q \\ q | x-a}} g'(q) = \sum_{\substack{q \leq Q \\ q | x-a}} g'(q) - \sum_{\substack{q \leq H \\ q | x-a}} g'(q) = g(x-a) - \sum_{\substack{q \leq H \\ q | x-a}} g'(q).$$

□

#### REFERENCES.

- [CL1] Coppola, G. and Laporta, M. - *Generations of correlation averages* - J. Numbers Volume 2014 (2014), Article ID 140840, 13 pages <http://dx.doi.org/10.1155/2014/140840> (see draft [arxiv:1205.1706v3](http://arxiv.org/abs/1205.1706v3) too)
- [CL2] Coppola, G. and Laporta, M. - *Sieve functions in arithmetic bands* - <http://arxiv.org/abs/1503.07502v3>
- [E] Elliott, P.D.T.A. - *On the correlation of multiplicative and the sum of additive arithmetic functions* Mem. Amer. Math. Soc. **112** (1994), no. 538, viii+88 pp. [MR95d:11099](#)

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